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On the use of Delaunay triangulation as 3D finite element modeler

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Delaunay triangulation, a geometric subdivision of any convex domain, is often used as a finite element modeling method, but there are still several problems, which originally come from the characteristics of Delaunay triangulation. One problem appears when we remove some nodes which are already introduced for the triangulation. In this case we aim to obtain the triangulation without nodes by partial modification of the Delaunay triangulation with the node. Another problem occurs when tetrahedra with zero volume are generated by Delaunay triangulation. In this case they must be removed for the numerical analysis in order to guarantee the numerical stability and good numerical solutions. In this paper these two problems occuring at the use of Delaunay triangulation are theoretically discussed.

Key words : Delaunay triangulation, Degeneracy, Tetrahedron, Automatic mesh generation

1. INTRODUCTION

According to the development of computer ability the finite element method is often used to solve mechanical phenomena in 3-dimensional space, and therefore, the finite element modeler is recognized more and more important. Many mesh generation methods have been already proposed, and they are classified into two groups; structured grid and unstructured grid. The former is mainly used for domains with rather simple geometry, but in case of analysis domain with complicated geometry the latter is used. In the sense, the latter is mainly used at the application of the finite element method even though the former can give better numerical results, since the latter can easily prepare a geometric model of any complicated boundary geometry. Especially so-called Delaunay triangulation has been introduced as a basic tool of 3D mesh generation methods [Boyer(1981), Watson(1981), Joe(1992)]. But, at the same time we know the triangulation may fail to give good finite element models since the method generates tetrahedral finite elements with zero volume or lacks the function to erase nodes used for the triangulation. In the former case the geometric model can't be used as a finite element model since the element with zero volume leads to the computational difficulty, and the latter case requires user to triangulate from the 1st to the last node once again and as a result it requires recalculation of the triangulation. Present paper shows some answers to these two problems, which occur at the application of Delaunay triangulation as a mesh generation.

2. DELAUNAY TRIANGULATION

Delaunay triangulation is a geometric subdivision of a space using a set of nodes. For the use of the triangulation we assume the position of nodes is given. As a result of the application of

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Delaunay triangulation, the convex space occupied by nodes is uniquely divided into a set of tetrahedra, whose circumsphere does not include any other node except four nodes forming each tetrahedron.

Assume that (n-1) nodes are already placed and introduced for Delaunay triangulation, and we aim to obtain Delaunay triangulation for n nodes from the result of (n-1)-node triangulation. The n-th triangulation is achieved as follows;

Introduce the n-th node, and select tetrahedron whose circumsphere includes the node. Gather these tetrahedra, and form a polyhedron by removing common triangles of gathered tetrahedra. Divide the polyhedron into a set of tetrahedra, each of which is formed using each triangular surface of the polyhedron and the n-th node. That, the polyhedron is divided into m tetrahedra for m-polyhedron, where "m" indicates the polyhedron is covered by m triangular surfaces.

This triangulation gives a unique subdivision for n nodes, but several solutions appear when more than five nodes locate on a surface of same sphere. This phenomenon is called "degeneracy". If 8 corner nodes of a cube are used for Delaunay triangulation, there are several solutions since 8 nodes locate on a same sphere.

In this paper we assume the triangulation, which is obtained first, is treated as the solution of Delaunay triangulation when "degeneracy" occurs. But, which triangulation is stored depends on the actual programming of Delaunay triangulation. Taniguchi & Fillion (1996) treats the case of degeneracy to adjust meshing on common surface of convex composite domain.

3. (n-1)-NODE DELAUNAY TRIANGULATION FROM n-NODE TRIANGULATION

Assume that a set of tetrahedra is already obtained by the use of Delaunay triangulation for n nodes. Our aim is to obtain Delaunay triangulation for (n-1) nodes from the result of n-node triangulation by removing a node, namely p(i), $1 \leqq i \leqq n$, which is arbitrarily selected among n nodes that are already used for Delaunay triangulation.

At the stage of the i-th Delaunay triangulation, a polyhedron is formed using a set of tetrahedra, whose circumspheres include the i-th node, and subdivided into new set of tetrahedra by connecting node i and each triangles which cover the surface of polyhedron.

According to the theory and the procedure of Delaunay triangulation the tetrahedra generated close to a node, namely p(i), are uniquely determined in general. But, in case of degeneracy how the tetrahedra are gathered are determined not only by the nodes of tetrahedra, whose circumspheres include p(i), but also their ordering to be introduced for the triangulation. Two dimensional explanation of the degeneracy is illustrated in Fig.1. In this example four nodes locate on a circle, and there are two triangulations both of which satisfy Delaunay triangulation. The degeneracy in case of 3-dimension occurs when more than 5 nodes locate on the surface of a same sphere,.e. several tetrahedra have a same circumsphere.

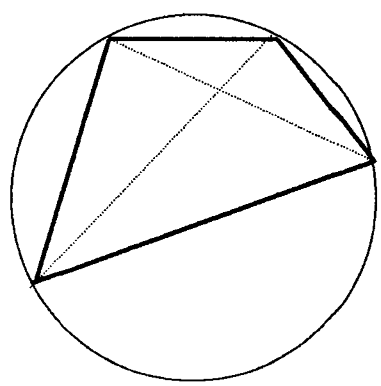

Fig.1 Example of degeneracy

From above consideration, the triangulation in case of degeneracy is normally determined by the programming technique, i.e. which triangulation among a set of possible triangulation is selected as a solution, since there are several solutions of triangulation in case of degeneracy..

In case of unique triangulation, all

tetrahedra, whose circumsphere include p(i) inside, are gathered and are assembled as a polyhedron by removing common triangles. And, the application of Delaunay triangulation to these nodes which appear on the surface of the polyhedron gives new triangulation without p(i). And, the replacement of these tetrahedra instead of tetrahera which are connected to p(i) in original triangulation gives the triangulation without p(i).

Now, we think of the case of degeneracy in detail.
Let

$$\{p(1),p(2), \dots , p(x), .., p(y), .., p(z),.., p(i), .., p(\alpha), .., p(\beta), .., p(n)\} \qquad (1)$$

be a set of n nodes which were already used for the triangulation, and we aim to delete p(i) among them and form a set of tetrahedra from the result of n-node Delaunay triangulation. From the tetrahedra whose circumspheres include p(i), we obtain a set of nodes, which is a portion of above node set, as following.

$$\{p(x), p(y), p(z), p(\alpha), p(\beta)\} \qquad (2)$$

In this case only 5 nodes in (2) contribute to form tetrahedra concerning p(i). But, in case of the degeneracy, in addition to these nodes, the triangulation surrounding p(i) is influenced by such nodes locating between p(1) and p(β) that locate on any ciucumsphere of a tetrahedron which are generated until the final triangulation surrounding p(i) is determined. It is explained using simple example shown in Fig. 2.

Fig. 2 shows a 3-dimensional configuration with 10 nodes, whose numbers show the ordering of nodes used for the triangulation. After the triangulation for these 10 nodes, we aim to delete the final node, p(10), in Fig. 2. p(10) indicated using * in Fig. 2 locates inside the cube.

Using the node p(10) we find out all tetrahedra which fulfil the volume surrounded by 8 nodes, i.e. p(1) through p(7) and p(9). On the contrary, if we apply Delaunay triangulation only for these 8 nodes except p(8), the generated tetrahedra fulfils the domain but sometime we find the different triangulation appears on the surface of the volume. This discrepancy is explained as following.

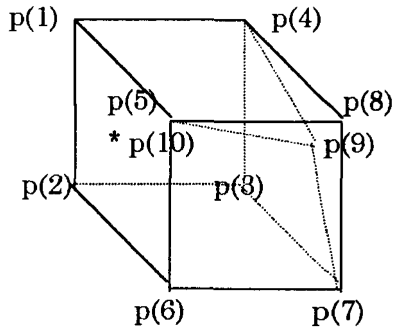

Fig. 2 Locations of 9 nodes

The set of nodes directly connected to p(i) after n-node triangulation is those which form tetrahedra after n-node triangulation, but during the triangulation other nodes also form tetrahedra with p(i). For example, after the triangulation using p(8), the triangulation of all surfaces of the cube of Fig.2 is already determined. Then, new node p(9) is introduced and the cube is subdivided into new set of tetrahedra. But, it should be noted that the triangulation using p(9) does not affect the triangulation of the surface of the cube of Fig.2, since p(9) is introduced after p(8). If the square, p(4), p(3), p(7) and p(8), is divided into two triangles using the diagonal, p(3)-p(8), the triangulation using p(9) can't divide the surface surrounded by four nodes, i.e., p(4), p(3), p(7) and p(8), into two triangles using the diagonal, p(4)-p(7). If this triangulation is accepted, two diagonals, p(3)-p(8) and p(4)-p(7), exist on a plane where a square surface locates.

If we find altogether three nodes, p(j), p(k) and p(m), each of which locates before p(i), between p(x) and p(i), and also between p(i) and p(β), respectively, the set of nodes, which are to be used to obtain (n-1)-node triangulation from n-node result, is

$$\{ p(j), p(x), p(y), p(k), p(z), p(\alpha), p(m), p(\beta)\} \quad (3)$$

As conclusions of this section the triangulation without p(i) from n-node Delaunay triangulation is obtained by considering following facts;

(1) all nodes which form tetrahedra with p(i),
(2) other nodes which locate on the surface of circumsphere of tetrahedron which appears at the application of Delaunay triangulation for the set of nodes that are connected to p(i), and
(3) the ordering of nodes, i.e. (1) and (2), which are used at the original triangulation.

4. REMOVAL OF TETRAHEDRON WITH ZERO VOLUME

After the application of Delaunay triangulation we sometimes find out the existence of such tetrahedra whose volume is zero. Such elements mainly occur on the surface of 3-dimensional volume when four nodes locate on a plane, but it may appear inside the volume. The existence of such tetrahedral finite elements leads to the failure of the numerical analysis, and, therefore, they must be removed the use of the numerical analysis.

Assume four nodes, P, A, B, and C in Fig. 3, form a tetrahedron with zero volume. Then, we find

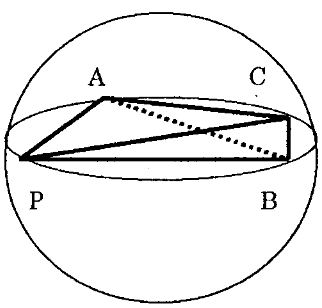

Fig. 3 Example of tetrahedraon with zero volume (Four nodes of tet(PBCA) locate on a plane.)

these four nodes locate on a surface of a sphere and also on a plane, since they must locate on a cirsumsphere of a tetrahedron. Fig. 3 shows the circumsphere of a tetrahedron, tet(PABC). Dotted and rigid lines in the figure show that these two edges of the tetrahedron cross on a plane each other.

We assume such tetrahedron is created inside a domain. Then, for two triangles on the front side of Fig.3 (upper half-space) there are two tetrahedra, i.e.,

tet(PACX) and tet(PBCY)

On the rear side (lower half-space) there are also two tetrahedra, i.e.,

tet(PABZ) and tet(ABCα)

In case of X=Y (and Z=α) two triangles locating on one side form only two tetrahedra which have the common top node.

We aim to delete tet(PABC) among generated tetrahedra. Then, the modification is required to delete the crossing of two lines on a same plane, A-B and P-C. The modification means to select only one line among two lines and also to modify tetrahedra concerning the removal of one line. It should be noticed that the modification of tetrahedra must done under the condition of Delaunay triangulation.

Four nodes, P, A, B, and C, locate on a circle, and they are in the degeneracy for 2-dimensional case. That is, both subdivisions using two diagonals of a quadrilateral, quad(PBCA), into triangles satisfy the condition of Delaunay triangulation.

(1) In case of Z=α (and / or X=Y)

In this case two triangles, tri(APB) and tri(ABC), have same top node and generate two tetrahedra, tet(PABZ) and tet(ABCZ). Then, the exchange of two diagonals from A-B to P-C leads to new set of tetrahedra, i.e., tet(PBCZ) and tet(PCAZ). And, the subdivision into new set is

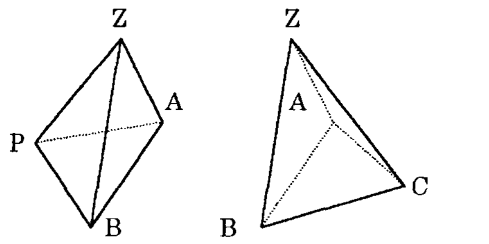

Fig.4-1 Before the exchange of diagonal

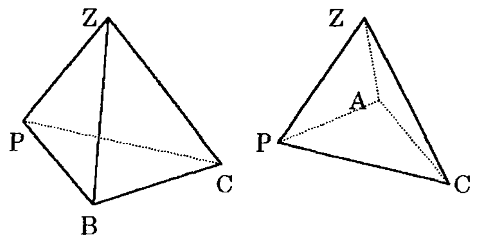

Fig.4-2 After the exchange of diagonal

obviously under the condition of Delaunay. triangulation. See Fig.4-1 and 4-2. It should be noted that five nodes, P, A, B, C, and Z, locate on a same sphere in this case. Same discussion is valid for the case of X=Y. Above discussion leads to the conclusion that two diagonals locating on a plane can be summarized into only one diagonal.

We can conclude that the tetrahedron with zero volume can be deleted if two triangles locating on, at least one side, of a plane have same top node.

(2) In case of $Z \neq \alpha$ (and $X \neq Y$)

Assume that four nodes, P, A, B and C, locate on a circle. Then, following discussion clarifies that the circumsphere of one tetrahedron, for example tet(PABZ), becomes the circumsphere of another tetrrahedron, namely tet(ABC α) as illustrated in Fig. 5. Moreover, following discussion clarifies that any tetrahedron locating between two triangles, namely tri(ABZ) and tri(AB α) of Fig.5, has also the same circumsphere.

Fig. 6 shows the relation between two spheres which have common circle. Radii of two spheres are expressed R and r (R > r). If surfaces of these two spheres are expressed as Sph(R) and Sph(r),

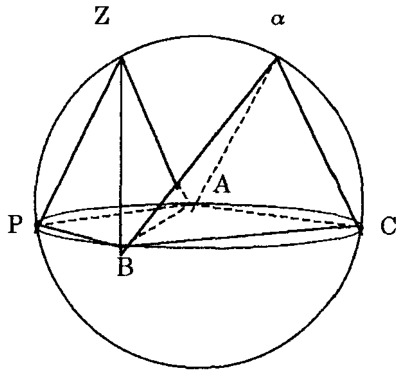

Fig.5 Circumsphere of two tetrahedra

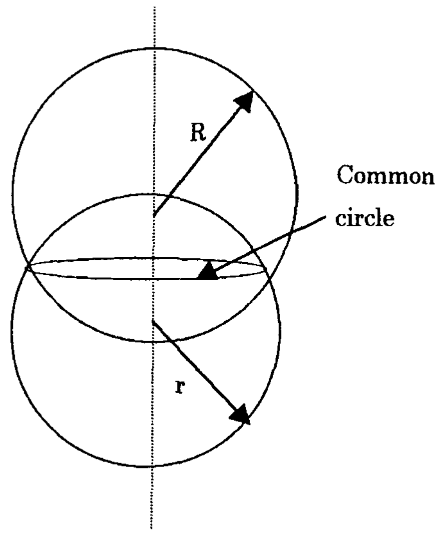

Fig.6 Relation between two shperes

respectively, following relations are obtained from the figure;

$$\mathrm{Sph(R)} \supseteq \mathrm{Sph(r)} \quad \text{above common circle}$$
$$\mathrm{Sph(R)} \subseteq \mathrm{Sph(r)} \quad \text{below common circle}$$

Then,

$$R = r$$

and, two top nodes, Z and α , of these two tetrahedra, tet(PABZ) and tet(ABC α), must locate on the same circumsphere as shown in Fig.5. The reason is explained as following; if $R \neq r$ and $R > r$, top node α of tet(ABC α) must locate inside the circumsphere of tet(PABZ). The tetrahedra treated there are generated using

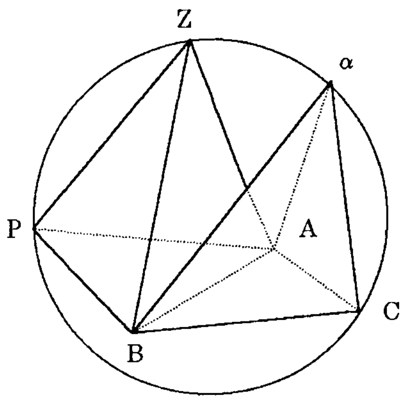

Fig.7-1 Before the exchange of diagonal

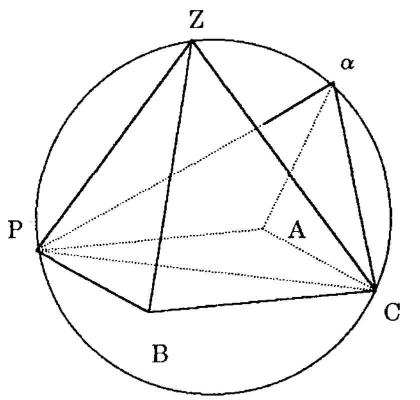

Fig.7-2 Aftrer he exchange of diagonal

Delaunay triangulation, and, therefore, this result clarifies the contradiction.

If Z $\neq$ α, there must be tetrahedra between two triangles, tri(ABZ) and tri(ABα), since the domain is divided into tetrahedra using Delaunay triangulation. Pick up a tetrahedron, namely tet(ABZγ), which is adjacent to tri(ABZ). Then, the node, γ, must locate on the circumsphere shown in Fig.5, since two tetrahedra, i.e., tet(PABZ) and tet(ABZγ), have common circle, and above discussion is valid for this case, too. Then, all tetrahedra locating between two triangles, namely tri(ABZ) and tri(ABα), must have same circumsphere.

Above discussion clarifies that each side of the tetrahedron with zero volume is surely divided into a set of tetrahedra all of which have only one circumsphere.

In this section we treat, at least, three tetrahedra exist in each side of quad(PBCA). In order to remove tet(ABCP) with zero volume, the exchange of diagonal in one side is required, and the exchange necessarily requires the subdivision of a polyhedron which are formed by the tetrahedra all of which have the same circumsphere.

Assume a polyhedron whose surface is covered by triangles, and we aim to divide the polyhedron into tetrahedra only using triangles on its surface. It should be noted that all nodes forming these triangles are on a same sphere. Simple example shown in Fig.7 clarifies that the exchange of diagonals requires the modification of other tetrahedra which are adjacent to those in concern. The exchange of diagonals necessarily changes the triangular surfaces of newly generated tetrahedra as shown in Fig.7. Therefore, the exchange of diagonals on quad(PBCA) requires the subdivision into new tetrahedra not only in the volume directly connected to quad(PBCA) but also to other area. Then we conclude that the removal of a tetrahedron with zero volume is impossible if there are more than two tetrahedra in both side of the tetrahedron with zero volume.

At the same time, above theoretical consideration indicates that the removal of zero-volume tetrahedron becomes possible if only two tetrahedra exist at least one side. Therefore, if more than two tetrahedra appear in both sides, nodes close to, at least, one side of the quad(PBCA) must be relocated as to generate only two tetrahedra in one side.

5. CONCLUDING REMARKS

In this paper the author theoretically studied on two basic problems concerning Delaunay triangulation which arise when it is used as the finite element modeling method, and the author could show following results:

(1) generally speaking, Delaunay triangulation for (n-1) nodes from the result of triangulation for n nodes is simply obtained from the latter by removing the node to be deleted and by replacing

with the result of the triangulation of nodes which are directly connected to the node, but,

(2) in case that the node to be deleted concerns with the case of degeneracy, (n-1)-node Delaunay triangulation can be obtained from the result of n-node triangulation by considering

① nodes directly connected to the deleted node,

② other nodes which locate on the surface of circumsphere of any tetrahedron that appears during the triangultion process for nodes of ①,

③ their ordering for the triangulation,

(3) the removal of tetrahedron with zero volume becomes possible when only two tetrahedra exist at least one side of the tetrahedron with zero volume, and

(4) for other cases except (3), the node relocation to appropriate position is required to satisfy the condition shown in (3), and then, the removal of tetrahedron with zero volume becomes possible.

These results can be introduced as efficient tools for the generation of finite element models which can give better numerical solutions. For example, the first two conclusions are effective to save CPU-time, since these conclusions indicate new programming technique to obtain Delaunay triangulation. The latter two conclusions show how to remove tetrahedra with zero volume, and the introduction of these conclusions into the mesh generation methods guarantees the mesh quality to certain extent. More improvement of mesh quality is expected by the introduction of Laplaican method, which relocates nodes to appropriate position.

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